ALGO

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SCIENCE TECHNOLOGY ENGINEERING MATHEMATICS

Appunti Trasformata di Laplace

UNI - Matematica

rev.0.1 - 05 set 2023

Draft version

Appunti in formato bozza, intesi esclusivamente di ausilio alle lezioni, che le integrano nelle descrizioni e nei ragionamenti su quanto viene riportato in queste pagine.

TRASFORMATA DI LAPLACE

Def Sia $f\ [0,+\infty] \to \mathbb{C}$

La trasformata di Laplace di f è

$$L(s) = \mathcal{L}[f](s) = \int_0^{+\infty} e^{-st} f(t)\,dt \quad \text{definita per } \operatorname{Re}(s) > \sigma_f$$

nuova variabile indipendente

$\mathcal{L}$ è un operatore lineare: $\mathcal{L}[\alpha f + \beta g](s) = \alpha \mathcal{L}[f] + \beta \mathcal{L}[g]$

primi esempi:

$$x(t) \xrightarrow{\mathcal{L}[x](s)} X(s) = \int_0^{+\infty} e^{-st} x(t)\,dt$$

$$\mathcal{L}[1] = \int_0^{+\infty} e^{-st}\,dt = \left.\frac{e^{-st}}{-s}\right]_0^{\infty} = 0 - \left(-\frac{1}{s}\right) = \frac{1}{s}$$

Integrazione per parti: $f = t$, $f' = 1$, $g = \frac{e^{-st}}{-s}$, $g' = e^{-st}$

$$\mathcal{L}[t] = \int_0^{+\infty} t\, e^{-st}\,dt = \left. \cancel{t \frac{e^{-st}}{-s}}\right]_0^{+\infty} - \int_0^{+\infty} \frac{e^{-st}}{-s} = \frac{1}{s}\int_0^{+\infty} e^{-st}\,dt = \frac{1}{s^2}$$

$\mathcal{L}[1] = \frac{1}{s}$

$$\mathcal{L}[x'(t)] = s X(s) - x(0)$$

ANTITRASFORMAZIONE

$$X(s) \xrightarrow{\mathcal{L}^{-1}} x(t) = \sum \operatorname{Res}\left(X(s)\, e^{st}, z_k\right)$$ metodo dei residui

$$x(t) \xleftarrow{\mathcal{L}} X(s) = \sum \frac{a_k}{s + b_k}$$ metodo del quoziente tra polinomi

APPLICAZIONE DI $\mathcal{L}$: Risoluzione eq. diff.

La trasformata di Laplace trasforma un problema differenziale in algebrico

$$y'(x)+y(x)=2x \xrightarrow[\text{notazione}]{\text{cambio}} x'(t)+x(t)=2t \xrightarrow{\mathcal{L}} sX(s)-x_0+X(s)=\frac{2}{s^2}$$

non è più eq. differenziale. Risolvendo:

$$(s+1)X(s)=x_0+\frac{2}{s^2} \Rightarrow X(s)=\frac{x_0}{(s+1)}+\frac{2}{s^2(s+1)}$$

torniamo alla notazione $x(t)$ antitrasformando

$$X(s)=\frac{x_0}{(s+1)}+\frac{2}{s^2(s+1)} \xrightarrow{\mathcal{L}^{-1}} x(t)=\sum \mathrm{Res}\left(X(s)\,e^{st}, z_k\right)$$

RESIDUI

SINGOLARITÀ DI $X(s)$

$s+1=0 \rightarrow s=-1$

$s^2=0 \rightarrow s=0$ doppio

$\{s=-1 \quad \begin{cases} s=-1 \\ s=0 \text{ doppio}\end{cases}$

$$\mathrm{Res}\left(\frac{x_0}{s+1}e^{st}, -1\right)=\left.\frac{x_0 e^{st}}{\cancel{s+1}}\right|_{s=-1}=x_0e^{-t}$$

$$\mathrm{Res}\left(\frac{2e^{st}}{s^2(s+1)}, 0\right)=\left.\frac{d}{ds}\left(\frac{2e^{st}}{\cancel{s^2}(s+1)}\right)\right|_{s=0}=\left.2\,\frac{te^{st}(s+1)-e^{st}}{(s+1)^2}\right|_{s=0}=2(t-1)$$

singolarità doppia

$$\mathrm{Res}\left(\frac{2e^{st}}{s^2(s+1)}, -1\right)=\left.\frac{2e^{st}}{s^2\cancel{(s+1)}}\right|_{s=-1}=2e^{-t}$$

$$x(t)=(x_0+2)e^{-t}+2t-2 \rightarrow y(x)=(y_0+2)e^{-x}+2x-2$$ soluzione generale

Assegnando una condizione per il calcolo di y_0 otteniamo una soluzione particolare (problema di Cauchy)

$$\begin{cases} y'(x)=-y(x)+x \\ y(0)=y_0 \end{cases} \Longrightarrow \text{soluzione particolare}$$

PROPRIETÀ

$$\mathcal{L}\left[e^{at}\right](s) = \frac{1}{s-a}$$

$$\mathcal{L}\left[\sin at\right](s) = \frac{a}{s^2+a^2}$$

$$\mathcal{L}\left[\cos(at)\right] = \frac{s}{s^2+a^2}$$

$$\mathcal{L}\left[\sinh(at)\right] = \frac{a}{s^2-a^2}$$

$$\mathcal{L}\left[\cosh(at)\right] = \frac{s}{s^2-a^2}$$

$$\mathcal{L}\left[t^n\right](s) = \frac{n!}{s^{n+1}}$$

TRASLAZIONE

$$\mathcal{L}\left[e^{at}\cdot f(t)\right] = F(s-a)$$

$$\mathcal{L}\left[u(t-a)f(t-a)\right](s) = e^{-as}F(s)$$

CAMBIO DI SCALA $a>0$

$$\mathcal{L}\left[f(at)\right] = \frac{1}{a}F\left(\frac{s}{a}\right)$$

Prodotto per t^n

$$\mathcal{L}\left[t^n f(t)\right] = (-1)^n \frac{d^n}{ds^n}\left(F(s)\right)$$

derivata n-esima

$$\mathcal{L}\left[f^{(n)}(x)\right] = s^n F(s) - \sum_{k=0}^{n-1} s^{n-1-k} f^{(k)}(0^+)$$

$$\mathcal{L}\left[f'(x)\right] = sF(s) - f(0)$$

$$\mathcal{L}\left[f''(x)\right] = s^2F(s) - sf(0) - f'(0)$$

Integrale

$$\mathcal{L}\left[\int_0^t f(x)\,dx\right] = \frac{F(s)}{s}$$

F. Periodica

$$\mathcal{L}\left[f(t)\right] = \frac{F_0(s)}{1-e^{-sT}}$$

F. GRADINO

$$\mathcal{L}\left[u(t)\right] = \mathcal{L}[1] = \frac{1}{s}$$

$$\mathcal{L}\left[u(t-a)\right] = \frac{1}{s-a}$$

δ DIRAC

$$\mathcal{L}\left[\delta(t)\right] = 1$$

CONVOLUZIONE

$$\mathcal{L}\left[f*g\right] = \mathcal{L}[f]\cdot\mathcal{L}[g]$$

QUESITO 4 (b)

Trasformata di Laplace della funzione

$$\begin{cases} y'' + 2y' - 3y = 4e^{3t} \\ y(0) = 10 \\ y'(0) = 6 \end{cases}$$

$$\mathcal{L}[y''] + 2\mathcal{L}[y'] - 3\mathcal{L}[y] = 4\mathcal{L}e^{3t}$$

$$s\mathcal{L}[y'] - y'(0) + 2\mathcal{L}[y'] - 3\mathcal{L}[y] = 4\mathcal{L}[e^{3t}]$$

$$(s+2)\mathcal{L}[y'] - y'(0) - 3\mathcal{L}[y] = 4\mathcal{L}[e^{3t}]$$

$$(s+2)\left(sF(s) - y(0)\right) - y'(0) - 3F(s) = \frac{4}{s-3}$$

$$(s^2+2s-3)F(s) - (s+2)\cdot 10 - 6 = \frac{4}{s-3}$$

$$(s-3)(s^2+2s-3)F(s) = 10(s-3)(s+2) + 6(s-3) + 4$$

$$F(s) = \frac{10(s-3)(s+2) + 6(s-3) + 4}{(s-3)(s^2+2s-3)} =$$

$$= \frac{10(s^2-s-6) + 6s - 18 + 4}{(s-3)(s^2+2s-3)} = \frac{10s^2 - 4s - 74}{(s-3)(s^2+2s-3)}$$

ESERCIZIO

Trasformata di Laplace della funzione

$$\begin{cases} y'' + y' + 2t + 1 = 0 \\ y(0) = 0 \\ y'(0) = 3 \end{cases}$$

$$\mathcal{L}[y''] + \mathcal{L}[y'] + \mathcal{L}[2t] + \mathcal{L}[1] = 0$$

$$s\mathcal{L}[y'] + y'(0) + \mathcal{L}[y'] + \frac{2}{s^2} + \frac{1}{s} = 0$$

Applico le proprietà:

$$\mathcal{L}[f'(x)] = sF(s) - f(0)$$

$$(s+1)\mathcal{L}(y') + y'(0) + \frac{2}{s^2} + \frac{1}{s} = 0$$

$$\left[(s+1)\left(sF(s) \cancel{- y(0)}\right) - 3 + \frac{2}{s^2} + \frac{1}{s} = 0\right] s^2$$

$$s^3(s+1)F(s) - 3s^2 + s + 2 = 0 \rightarrow F(s) = \frac{3s^2 - s - 2}{s^3(s+1)}$$

ESERCIZIO

Trasformata di Laplace della funzione

$$x(t) = 5t^2 e^{3t}$$

$$\mathcal{L}\left[5t^2 e^{3t}\right] = 5\frac{2!}{(s-3)^3} = \frac{10}{(s-3)^3}$$

Applico le proprietà:

$$\mathcal{L}\left[e^{at}\right](s) = \frac{1}{s-a}$$

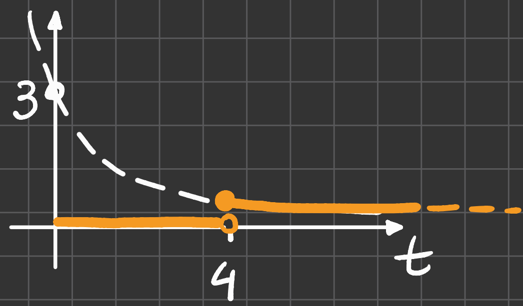
$$\mathcal{L}\left[t^n\right](s) = \frac{n!}{s^{n+1}}$$

ESERCIZIO

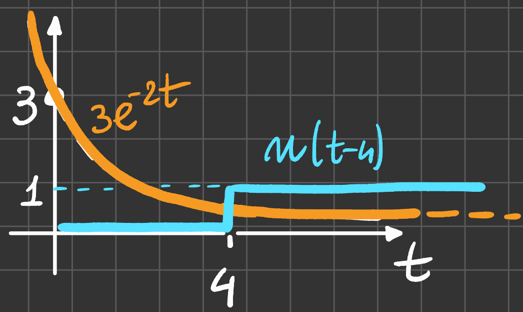
Trasformata di Laplace della funzione

$$x(t) = \begin{cases} 0 & \text{se} \quad 0<t<4 \\ 3e^{-2t} & \text{se} \quad t\geq 4 \end{cases}$$

$$x(t) = \begin{cases} 0 & \text{se} \quad 0<t<4 \\ 3e^{-2t} & \text{se} \quad t\geq 4 \end{cases}$$



$$x(t) = 3e^{-2t}\left(u(t-4)\right)$$



$$\mathcal{L}\left[3e^{-2(t-4+4)}u(t-4)\right] =$$

Applico la proprietà:

1) $\mathcal{L}\left[u(t-a)f(t-a)\right] e^{-as}F(s)$

$$= 3e^{-8}\mathcal{L}\left[e^{-2(t-4)}u(t-4)\right] =$$

$$= 3e^{-8}e^{-4s}\cdot\mathcal{L}\left[e^{-2t}\right] =$$

Applico la proprietà:

$$\mathcal{L}\left[e^{at}\right](s) = \frac{1}{s-a}$$

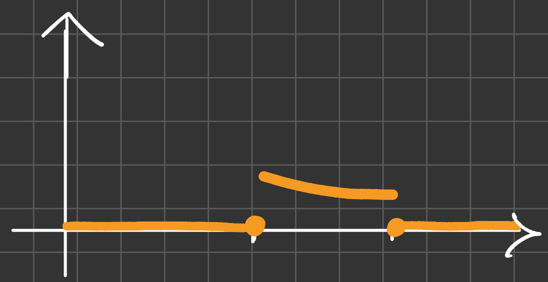
$$= 3e^{-8}\cdot e^{-4s}\cdot\frac{1}{s+2} =$$

$$= \frac{3e^{-4(s+2)}}{s+2} \quad \text{per } s>2$$

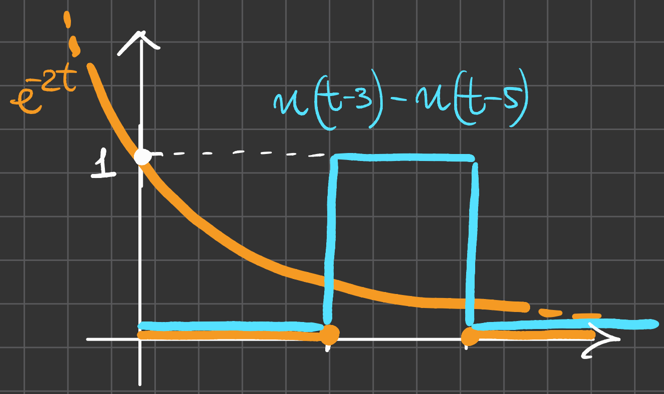
ESERCIZIO

Trasformata di Laplace della funzione

$$x(t) = \begin{cases} e^{-2t} & \text{se } 3<t<5 \\ 0 & \text{altrimenti} \end{cases}$$



$$x(t) = \begin{cases} e^{-2t} & \text{se } 3<t<5 \\ 0 & \text{altrimenti} \end{cases}$$



$$x(t) = e^{-2t}\left(u(t-3)-u(t-5)\right)$$

$$\mathcal{L}[x(t)] = \mathcal{L}\left[e^{-2(t-3+3)}u(t-3)\right] - \mathcal{L}\left[e^{-2(t-5+5)}u(t-5)\right]$$

Applico la proprietà:

$$\mathcal{L}[u(t-a)f(t-a)] = e^{-as}F(s)$$

$$= e^{-6}\mathcal{L}\left[e^{-2(t-3)}u(t-3)\right] - e^{-10}\mathcal{L}\left[e^{-2(t-5)}u(t-5)\right] =$$

$$\downarrow \quad \frac{e^{-3s}}{s+2} \qquad\qquad \downarrow \quad \frac{e^{-5s}}{s+5}$$

Applico la proprietà:

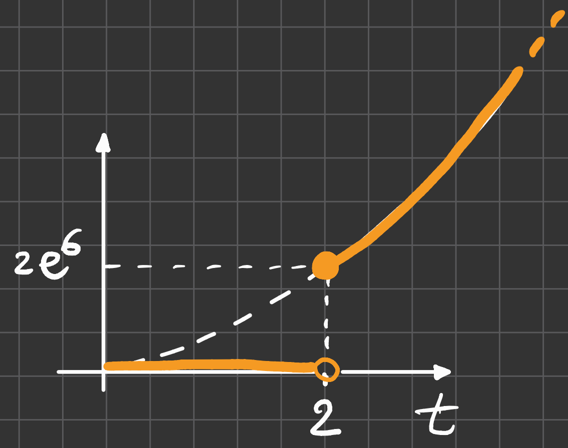
$$\mathcal{L}[u(t-a)f(t-a)] = e^{-as}F(s)$$

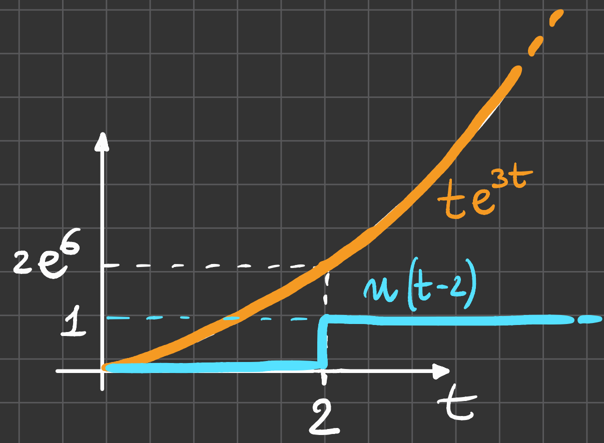
$$\mathcal{L}\left[e^{at}\right](s) = \frac{1}{s-a}$$

$$= \frac{e^{-6}\cdot e^{-3s} - e^{-10}e^{-5s}}{s+2} = \frac{e^{-6-3s} - e^{-10-5s}}{s+2} = \frac{e^{-3(2+s)} - e^{-5(2+s)}}{s+2}$$

ESERCIZIO

$$x(t) = \begin{cases} 0 & \text{se } 0 < t \leq 2 \\ t e^{3t} & \text{se } t > 2 \end{cases}$$





$$x(t) = te^{3t} u(t-2)$$

$$\mathcal{L}[x(t)] = \mathcal{L}[t\, e^{3t} u(t-2)] =$$

Applico le proprietà:
$\mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}(F(s))$

$$= (-1)\frac{d}{ds}\mathcal{L}\left[e^{3(t-2+2)} u(t-2)\right] =$$

Applico le proprietà:
1) $\mathcal{L}[u(t-a) f(t-a)] = e^{-as} F(s)$

$$= -e^6 \frac{d}{ds}\left(e^{-2s} \cdot \frac{1}{s-3}\right) = +e^6 \, \frac{+2e^{-2s}(s-3) + e^{-2s}}{(s-3)^2} =$$

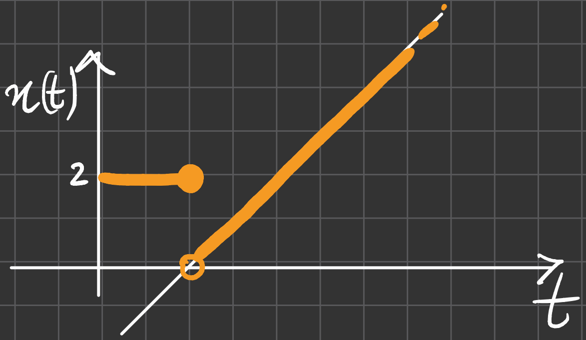
$$= e^6 \, \frac{e^{-2s}(2s-6+1)}{(s-3)^2} = \frac{(2\cdot s - 5)e^{6-2s}}{(s-3)^2}$$

ESERCIZIO

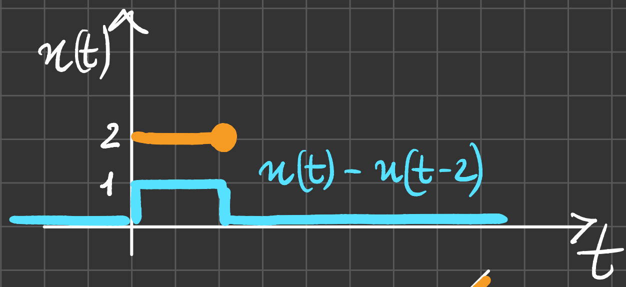
Trasformata di Laplace della funzione

$$x(t) = \begin{cases} 2 & 0 < t \leq 2 \\ t-2 & \text{se } t > 2 \end{cases}$$

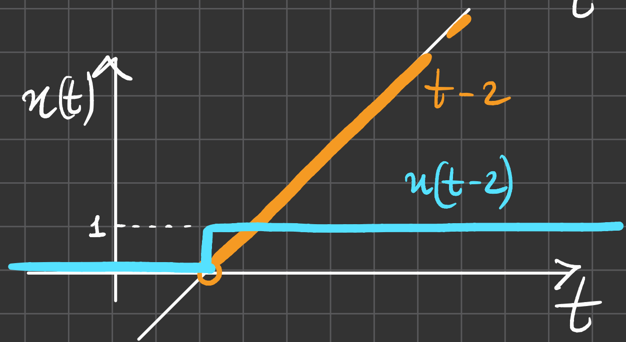
$$x(t) = \begin{cases} 2 & 0 < t \leq 2 \\ t-2 & \text{se } t > 2 \end{cases}$$



$$2\left[u(t) - u(t-2)\right]$$



$$(t-2)\left[u(t-2)\right]$$



$$x(t) = 2\left[u(t) - u(t-2)\right] + (t-2)\left[u(t-2)\right]$$

$$\mathcal{L}\left[x(t)\right] = 2\mathcal{L}\left[u(t)\right] - 2\mathcal{L}\left[u(t-2)\right] + \mathcal{L}\left[(t-2)u(t-2)\right]$$

$$= \frac{2}{s} - \frac{2e^{-2s}}{s} + \frac{e^{-2s}}{s^2} =$$

$$= \frac{2}{s}\left(1 - e^{-2s}\right) + \frac{e^{-2s}}{s^2}$$

$s > 0$ $\quad$ $s > 0$ $\quad$ per $s > 0$

Applico le proprietà:

$$\mathcal{L}\left[u(t)\right] = \mathcal{L}\left[1\right] = \frac{1}{s}$$

$$\mathcal{L}\left[u(t-a)f(t-a)\right] = e^{-as}F(s)$$

ESERCIZIO

Trasformata di Laplace della funzione

$$x(t) = \int_0^t r e^{-r} \sinh(3r)\, dr$$

$$x(t) = \int_0^t r e^{-r} \sinh(3r)\, dr$$

Applico la proprietà

$$\mathcal{L}\left[\int_0^t f(x)\, dx\right] = \frac{F(s)}{s}$$

$$\mathcal{L}[x(t)] = \frac{\mathcal{L}[r e^{-r} \sinh(3r)]}{s} =$$

Applico la proprietà:

$$\mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}(F(s))$$

$$= \frac{-\frac{d}{ds}\mathcal{L}[e^{-r}\sinh(3r)]}{s} =$$

Applico la proprietà

$$\mathcal{L}[\sinh(at)] = \frac{a}{s^2-a^2}$$

$$\mathcal{L}[e^{at} \cdot f(t)] = F(s-a)$$

$$= \frac{-\frac{d}{ds}\left(\frac{3}{(s+1)^2-3^2}\right)}{s} =$$

$$= \frac{-\frac{-3\cdot 2(s+1)}{[(s+1)^2-9]^2}}{s} = \frac{6(s+1)}{s[(s+1)^2-9]^2}$$

ESERCIZIO

Trasformata di Laplace dell'eq. diff.

$$\begin{cases} x'' - 4x' + 5x = e^{2t}\sin t \\ x(0) = 0 \\ x'(0) = 3 \end{cases}$$

Applico le proprietà: $\mathcal{L}[f''(x)] = s^2F(s) - sf(0) - f'(0)$

Applico le proprietà: $\mathcal{L}[f'(x)] = sF(s) - f(0)$

Applico le proprietà: $\mathcal{L}[\sin(at)] = \frac{a}{s^2+a^2}$ $\quad \mathcal{L}[e^{at}\cdot f(t)] = F(s-a)$

$$\mathcal{L}[x''] - 4\mathcal{L}[x'] + 5F(s) = \mathcal{L}[e^{2t}\sin t]$$

$$s^2F(s) - s\underbrace{\cancel{f(0)}}_{0} - f'(0) \qquad sF(s) - \underbrace{\cancel{f(0)}}_{0} \qquad \frac{1}{(s-2)^2+1}$$

$$F(s)\left(s^2 - 4s + 5\right) - 3 = \frac{1}{(s-2)^2+1}$$

$$F(s)\left(s^2 - 4\cdot s + 5\right) = \frac{1}{s^2 - 4s + 4 + 1} + 3$$

$$F(s)\left(s^2 - 4s + 5\right) = \frac{1 + 3s^2 - 12s + 15}{s^2 - 4s + 5}$$

$$F(s) = \frac{3s^2 - 12s + 16}{(s^2 - 4s + 5)^2}$$

$$\frac{\Delta}{4} = \left(\frac{b}{2}\right)^2 - ac = 4 - 5 = -1$$

$s = \frac{2 \pm \sqrt{-1}}{1} = 2 \pm i = a$ ascissa di convergenza

$Re(s) > 2$ Dominio di convergenza

ESERCIZIO

Antitrasformata di $\dfrac{4-2s}{s^2(s+2)}$

$s=-2$ semplice

$s=0$ doppio

$$\mathrm{Res}\left(\frac{(4-2s)e^{st}}{s^2(s+2)},0\right)=\frac{d}{ds}\left(\frac{2(2-s)e^{st}}{\cancel{s^2}(s+2)}\right)\Bigg|_{s=0}=$$

formula di derivazione per prodotti multipli

$$=\frac{2(2-s)e^{st}}{s+2}\cdot\left(\frac{-1}{2-s}+\frac{t\cancel{e^{st}}}{\cancel{e^{st}}}-\frac{1}{s+2}\right)\Bigg|_{s=0}=\frac{4}{2}\cdot\left(-\frac{1}{2}+t-\frac{1}{2}\right)=2(t-1)$$

$$\mathrm{Res}\left(\frac{(4-2s)e^{st}}{s^2(s+2)},-2\right)=\frac{(4-2s)e^{st}}{s^2\cancel{(s+2)}}\Bigg|_{s=-2}=\frac{8\cdot e^{-2t}}{4}=2e^{-2t}$$

$$x(t)=2e^{-2t}+2t-2$$

ESERCIZIO

Antitrasformata di $\dfrac{2s^2-4s^3-3}{s^2(s^2+1)}$

$s^2 \rightarrow s=0$ doppio

$(s^2+1) \rightarrow (s+i)(s-i)=0 \begin{cases} s=-i & \text{singolo} \\ s=i & \text{singolo} \end{cases}$

singolarità doppia

$$\operatorname{Res}\left(\frac{(2s^2-4s^3-3)e^{st}}{s^2(s^2+1)}, 0\right) = \frac{d}{ds}\left(\frac{(2s^2-4s^3-3)e^{st}}{\cancel{s^2}(s^2+1)}\right)\Bigg|_{s=0} =$$

formula di derivazione "produttoria"

$$= \frac{(2s^2-4s^3-3)e^{st}}{s^2(s^2+1)} \cdot \left(\frac{4s-12s^2}{2s^2-4s^3-3} + \frac{t\cancel{e^{st}}}{\cancel{e^{st}}} - \frac{2s}{s^2+1}\right)\Bigg|_{s=0} = 3\cdot(0+t-0) = 3t$$

$$\operatorname{Res}\left(\frac{(2s^2-4s^3-3)e^{st}}{s^2(s^2+1)}, i\right) = \frac{(2s^2-4s^3-3)e^{st}}{s^2(s+i)\cancel{(s-i)}}\Bigg|_{s=i} = \frac{2i^2-4i^3-3}{i^2(2i)}e^{it} =$$

$$= \frac{-2+4i-3}{-2i}e^{it} = \frac{-5+4i}{-2i}e^{it} = \frac{5}{2i}e^{it} - 2e^{it}$$

$$\operatorname{Res}\left(\frac{(2s^2-4s^3-3)e^{st}}{s^2(s^2+1)}, -i\right) = \frac{(2s^2-4s^3-3)e^{st}}{s^2\cancel{(s+i)}(s-i)}\Bigg|_{s=-i} = \frac{2i^2+4i^3-3}{i^2(-2i)}e^{-it} =$$

$$= \frac{-2-4i-3}{2i}e^{-it} = \frac{-5-4i}{2i}e^{-it} = -\frac{5}{2i}e^{-it} - 2e^{-it}$$

$$x(t) = t + \frac{5}{2i}e^{it} - 2e^{it} - \frac{5}{2i}e^{-it} - 2e^{-it} = t + 5\Bigg(\underbrace{\frac{e^{it}-e^{-it}}{2i}}_{\sin(t)}\Bigg) - 2\Big(\underbrace{e^{it}+e^{-it}}_{2\cos(t)}\Big)$$

$$= 3t + 5\cos(t) - 4\sin(t)$$

$$\mathcal{L}[e^{at}](s) = \frac{1}{s-a}$$

$$\mathcal{L}[\sin at](s) = \frac{a}{s^2+a^2}$$

$$\mathcal{L}[\cos(at)] = \frac{s}{s^2+a^2}$$

$$\mathcal{L}[\sinh(at)] = \frac{a}{s^2-a^2}$$

$$\mathcal{L}[\cosh(at)] = \frac{s}{s^2-a^2}$$

$$\mathcal{L}[t^n](s) = \frac{n!}{s^{n+1}}$$

TRASLAZIONE

$$\mathcal{L}[e^{at}\cdot f(t)] = F(s-a)$$

$$\mathcal{L}[u(t-a)f(t-a)](s) = e^{-as}\,F(s)$$

CAMBIO DI SCALA $a>0$

$$\mathcal{L}[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right)$$

Prodotto per t^n

$$\mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}(F(s))$$

derivata n-esima

$$\mathcal{L}[f^{(n)}(x)] = s^n F(s) - \sum_{k=0}^{n-1} s^{n-1-k} f^{(k)}(0^+)$$

$$\mathcal{L}[f'(x)] = sF(s) - f(0)$$

$$\mathcal{L}[f''(x)] = s^2F(s) - sf(0) - f'(0)$$

Integrale

$$\mathcal{L}\left[\int_0^t f(x)\,dx\right] = \frac{F(s)}{s}$$

F. Periodica

$$\mathcal{L}[f(t)] = \frac{F_0(s)}{1-e^{-sT}}$$

F. GRADINO

$$\mathcal{L}[u(t)] = \mathcal{L}[1] = \frac{1}{s}$$

δ DIRAC

$$\mathcal{L}[\delta(t)] = 1$$

CONVOLUZIONE

$$\mathcal{L}[f*g] = \mathcal{L}[f]\cdot\mathcal{L}[g]$$

FORMULE UTILI

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$$

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2}$$

$$\frac{d}{ds}\left(\frac{f_1 \cdot f_2 \cdot f_3}{g_1 \cdot g_2 \cdot g_3}\right)\Bigg|_{s=s_0} = \frac{f_1 \cdot f_2 \cdot f_3}{g_1 \cdot g_2 \cdot g_3}\Bigg|_{s=s_0} \cdot \left(\frac{f_1'}{f_1} + \frac{f_2'}{f_2} + \frac{f_3'}{f_3} - \frac{g_1'}{g_1} - \frac{g_2'}{g_2} - \frac{g_3'}{g_3}\right)\Bigg|_{s=s_0}$$

ESERCIZI

$$\begin{cases} x'' - 2x' - 8x = e^{4t} \\ x(0) = 0 \qquad x'(0) = 0 \end{cases}$$

$$\mathcal{L}[x'' - 2x' - 8x] = \mathcal{L}[e^{4t}]$$

$$\mathcal{L}[x''] - 2\mathcal{L}[x'] - 8\mathcal{L}[x] = \mathcal{L}[e^{4t}]$$

$$s\mathcal{L}[x'] - x'_0 \qquad s F(s) - \cancel{x_0} \; 0 \qquad F(s) \qquad \frac{1}{s-4}$$

$$s\left(s F(s) - \cancel{x_0'} - \cancel{x_0}\right) \quad 0 \quad 0$$

$$s^2 F(s) - 2sF(s) - 8F(s) = \frac{1}{s-4}$$

$$F(s)\left(\underbrace{s^2 - 2s - 8}_{(s+2)(s-4)}\right) = \frac{1}{(s-4)} \rightarrow \boxed{F(s) = \frac{1}{(s+2)(s-4)^2}}$$

Scomponendo in fratti semplici:

$$\frac{A}{s+2} + \frac{B}{(s-4)} + \frac{C}{(s-4)^2}$$

$$\frac{A(s-4)^2+B(s+2)(s-4)+C(s+2)}{(s+2)(s-4)^2}=$$

$$=\frac{As^2-8As+16A+Bs^2-2Bs-8B+Cs+2C}{(s+2)(s-4)^2}=$$

$$=\frac{s^2(A+B)+s(-8A-2B+C)+(16A-8B+2C)}{(s+2)(s-4)^2}$$

$$\begin{cases} A+B=0 \longrightarrow A=-B \\ C-8A-2B=0 \longrightarrow C=8A+2B=8A-2A=6A \\ 8A-4B+C=1 \longrightarrow C=4B-8A+1 \longrightarrow 6A=-12A+1 \longrightarrow A=\frac{1}{18} \end{cases}$$

$$6A \quad -4A$$

$$A=\frac{1}{18}$$

$$B=-A=-\frac{1}{18}$$

$$C=-12A+1=\frac{12}{18}+1=\frac{30}{18}=\frac{15}{9}$$

$$\frac{1}{36(s+2)}+\frac{10-s}{36(s-4)^2}=$$

$$\frac{1}{36(s+2)}-\frac{1}{36(s-4)}+\frac{1}{6(s-4)^2}$$

$$\frac{1}{36}\mathcal{L}^{-1}\left[\frac{1}{(s+2)}\right]-\frac{1}{36}\mathcal{L}^{-1}\left[\frac{1}{s-4}\right]+\frac{1}{6}\mathcal{L}^{-1}(s-4)^2$$

$$x(t)=\frac{1}{36}e^{-2t}-\frac{1}{36}e^{4t}+\frac{1}{6}te^{4t}$$

QUESITO 3 (a)?

$$x(t) = (e^{3t} + \cos t)^2$$

$$\mathcal{L}[(e^{3t}+\cos t)^2] = \mathcal{L}[e^{6t}] + 2\mathcal{L}[e^{3t}\cos t] + \mathcal{L}[\underbrace{\cos^2 t}_{\frac{\cos 2t+1}{2}}] =$$

$$= \mathcal{L}[e^{6t}] + 2\mathcal{L}[e^{3t}\cos t] + \frac{1}{2}\mathcal{L}[\cos(2t)] + \frac{1}{2}\mathcal{L}[1]$$

$$\downarrow \quad \frac{1}{s-6} \qquad \downarrow \quad \frac{s-3}{1+(s-3)^2} \qquad \downarrow \quad \frac{1}{2}\,\frac{s}{4+s^2} \qquad \downarrow \quad \frac{1}{s}$$

$s-6>0$

$s>6$

$s^2-6s+9+1>0$

$s^2-6s+10>0$

$s=\frac{3\pm\sqrt{9-10}}{2}$

$(s+2)(s-2)>0$

$s<-2 \lor s>2$

$$= \frac{1}{s-6} + \frac{2(s-3)}{1+(s-3)^2} + \frac{s}{4(s^2+4)} + \frac{1}{2s}$$

$$\frac{s^2+2s^2+8}{4s(s^2+4)} = \frac{3s^2+8}{4s(s^2+4)}$$